A Rich Collection of Squashing Functions

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The Fuzzy Thought Amplifier (FTA) is a tool for constructing and exercising fuzzy cognitive maps (FCM). The FCM squashing function reduces the summed input event values to the dynamic range of the state. The classical squashing function is the logistic function. In the FTA many other forms are available: arc tangent, multiple step, expanding, etc. This rich set offers tools for a more stimulating environment for the thought modeling purpose of fuzzy cognitive maps. The discussion will demonstrate the 20 or so varieties of FTA squashing functions and invite ideas as to the potential meaning and use of these constructs.

- **Fuzzy Cognitive Maps**
- **Conceptual State: activation**
- **Causal Event: weight**
- **multiply and summing process**
- **StateCount**
- **squashing operation**
- **look up table, fuzzy estimation or mathematical function**

Fuzzy Cognitive Maps

Each Conceptual State activation is the result of taking all the Causal Event weights pointing into the State, multiplying each weight by the State activation from whence the Event comes, summing all the results of these multiplications, and then squashing the results of the summing so it is between 0 and +1 (or 0 to 100%) or -1 and +1 (-100% to +100)..

This multiply and summing process is a linear operation. Being linear all the operations and results are easily seen from the laws of mathematics. The results lies within the range of -StateCount to +StateCount. Where StateCount is the number of Conceptual States in the Fuzzy Cognitive Map project.

The squashing operation is nonlinear. The operation could be arbitrary through a look up table or a fuzzy estimation surface. The Fuzzy Thought Amplifier uses a collection mathematically defined squashing functions. Of these the logistic function is the one that has been classically used. . This function symmetrically around zero on the input side and around .5 on the output converts the input of -StateCount to +StateCount to the output of 0.0 to 1.0. The squashing function output value defines the Conceptual State activation to which it is directed.

Here is the squashing function equation:

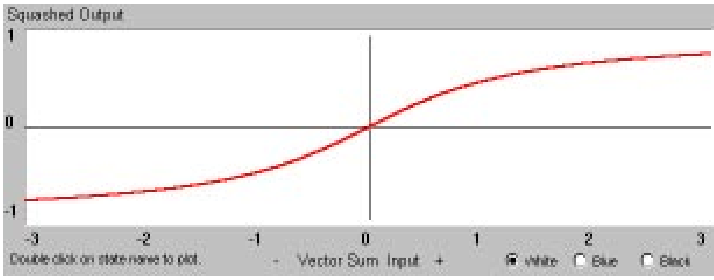
$$\text{Activation} = \text{Squash}(X)$$

$$X = (\text{Summation} + \text{Bias}) \text{ x Gai}$$

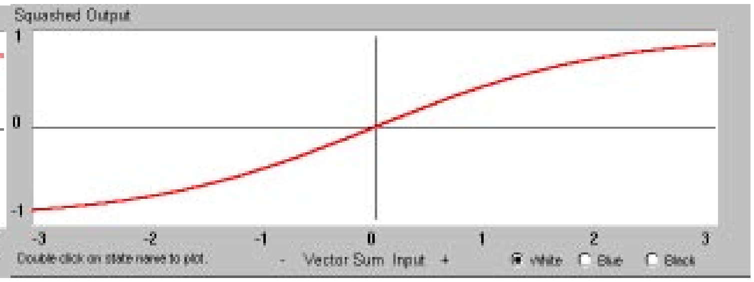
This produces a shaped transform where Summation(S) plus the Bias(B) times Gain(G) is the input and Activation(A) is the output. In addition each squashing shape may be expanded to an output that ranges from -1 to +1 (bipolar) and the input may be negated and the output may be inverted by subtracting from minus one.

Here are pictures of the basic shapes with the bipolar variation. Here also are the unipolar equations of the various inference function shapes(bipolar as well for the first two functions). The bipolar equations are similar but scaled and offset to center around 0.0. Inversion inverts the high and low limits.. Negation negates the equation output.

These pictures are with a Gain of 1,0 and a Bias of 0.0. The Gain will affect the scaling and Bias will affect the offset. With all the variation there are 144 shapes. Because of equivalence only 112 are unique. With Gain and Bias control the individual State squashing functions are quite flexible.



ArcTangent



Logistic Function:

Unipolar:

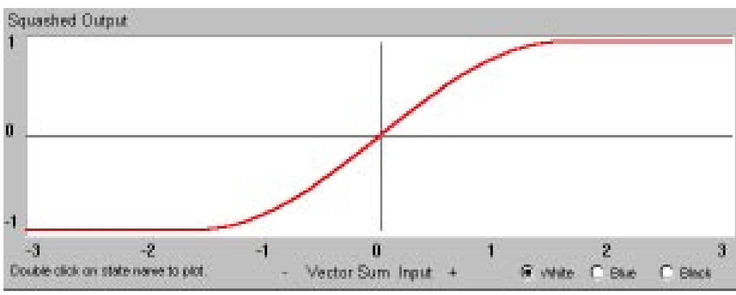
$$A = \tan^{-1}\left(\frac{(S+B)G}{\pi} + 0.5\right)$$

$$A = \frac{1}{1+e^{-((S+B)G)}}$$

Bipolar:

$$A = 2\tan^{-1}\left(\frac{(S+B)G}{\pi}\right)$$

$$A = \frac{2}{1+e^{-((S+B)G)}} - 1$$

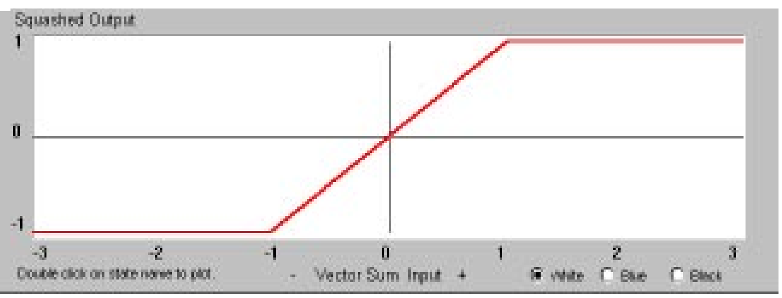


Sine Function:

$$A = 1.0 \text{ if } (S+B)G > .5\pi$$

$$A = 0.0 \text{ if } (S+B)G < -.5\pi$$

$$A = \sin((S+B)G) + 0.5 \text{ otherwise}$$

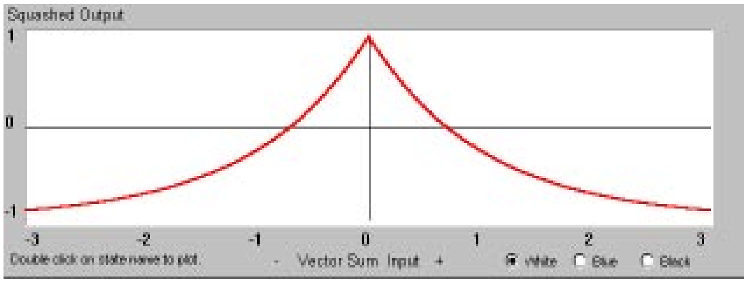


Linear Function:

$$A = 1.0 \text{ if } (S+B)G > 1.0$$

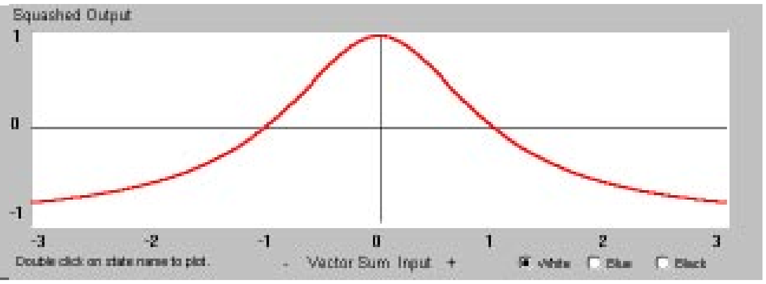
$$A = 0.0 \text{ if } (S+B)G < 0.0$$

$$A = (S+B)G \text{ otherwise}$$



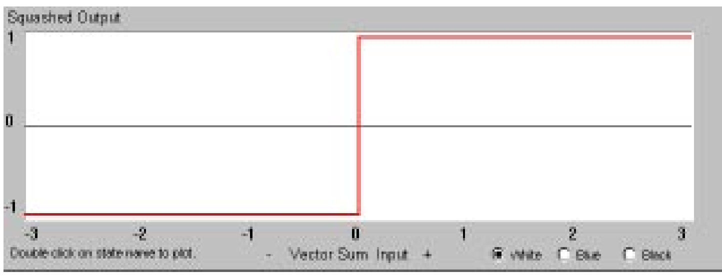
Gaussian Function:

$$A = e^{-\left|((S+B)G)\right|}$$



Cauchy Function:

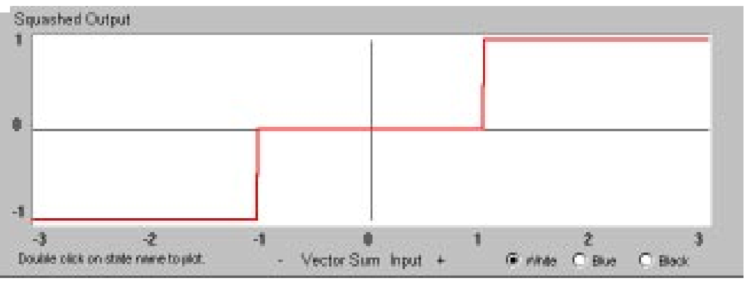
$$A = \frac{1}{1 + \left((S+B)G\right)^2}$$



Binary Function:

$$A = 1.0 \text{ if } (S+B)G > 0.0$$

$$A = 0.0 \text{ if } (S+B)G < 0.0$$

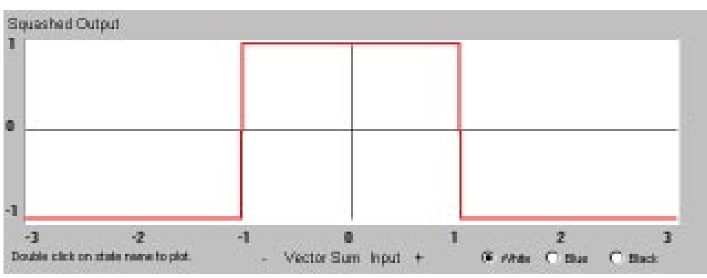


Trinary Function:

$$A = 1.0 \text{ if } (S+B)G \geq 1,0$$

$$A = 0.5 \text{ if } (S+B)G < 1,0 \text{ and } > -1.0$$

$$A = 0.0 \text{ if } (S+B)G \leq -1.0$$

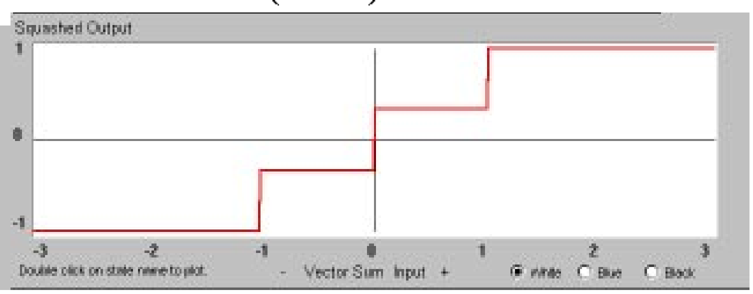


Slot Function:

$$A = 0.0 \text{ if } (S+B)G \geq 1.0$$

$$A = 1.0 \text{ if } (S+B)G < 1.0 \text{ and } > -1.0$$

$$A = 0.0 \text{ if } (S+B)G \leq -1.0$$



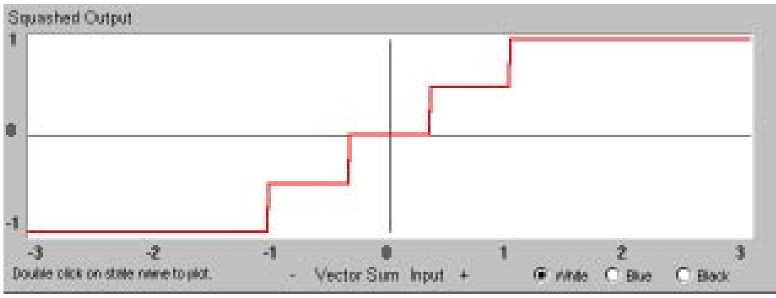
Quadary Function

$$A = 1.0 \text{ if } (S+B)G \geq 1.0$$

$$A = 0.66 \text{ if } (S+B)G < 1.0 \text{ and } \geq 0.0$$

$$A = 0.33 \text{ if } (S+B)G < 0.0 \text{ and } \geq -1.0$$

$$A = 0.0 \text{ if } (S+B)G \leq -1.0$$



Pentary Function:

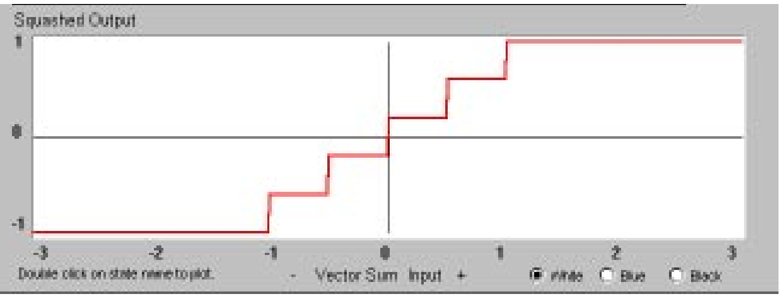
$A = 1.0$ if $(S + B)G \geq 1.0$

$A = 0.75$ if $(S + B)G < 1.0$ and > 0.333

$A = 0.5$ if $(S + B)G < 0.333$ and $> -.333$

$A = 0.25$ if $(S + B)G < -0.333$ and > -1.0

$A = 0.0$ if $(S + B)G \leq -1.0$



Hexary Function:

$A = 1.0$ if $(S + B)G > 1.0$

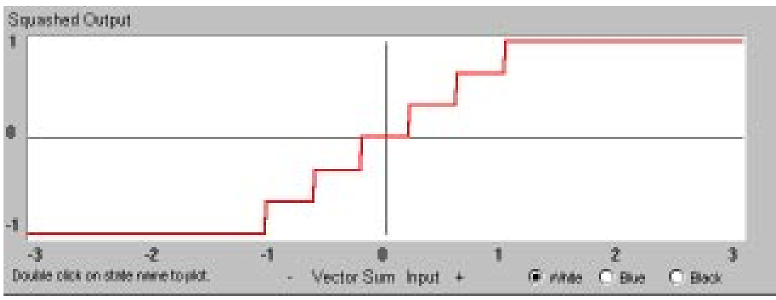
$A = 0.8$ if $(S + B)G < 1.0$ and ≥ 0.5

$A = 0.6$ if $(S + B)G < 0.5$ and ≥ 0.0

$A = 0.4$ if $(S + B)G < 0.0$ and ≥ -0.5

$A = 0.2$ if $(S + B)G < -0.5$ and > -1.0

$A = 0.0$ if $(S + B)G < -1.0$



Septary Function:

$A = 1.0$ if $(S + B)G \geq 1.0$

$A = 0.825$ if $(S + B)G < 1.0$ and $\geq .6$

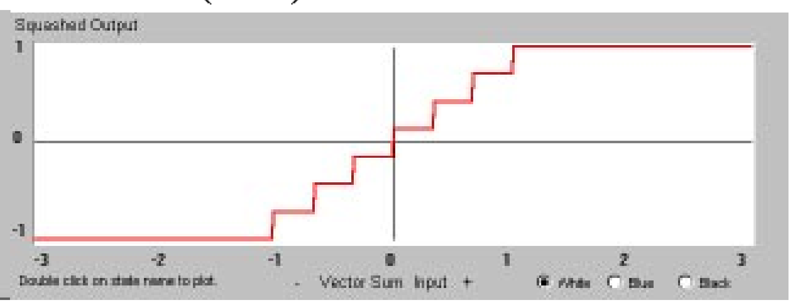
$A = 0.66$ if $(S + B)G < 0.6$ and $\geq .2$

$A = 0.5$ if $(S + B)G < 0.2$ and ≥ -0.2

$A = 0.33$ if $(S + B)G < -0.2$ and ≥ -0.6

$A = 0.165$ if $(S + B)G < -0.6$ and > -1.0

$A = 0.0$ if $(S + B)G < -1.0$



Octary Function:

$A = 1.0$ if $(S + B)G \geq 1.0$

$A = 6/7$ if $(S + B)G < 1.0$ and $\geq .66$

$A = 5/7$ if $(S + B)G < 0.66$ and ≥ 33

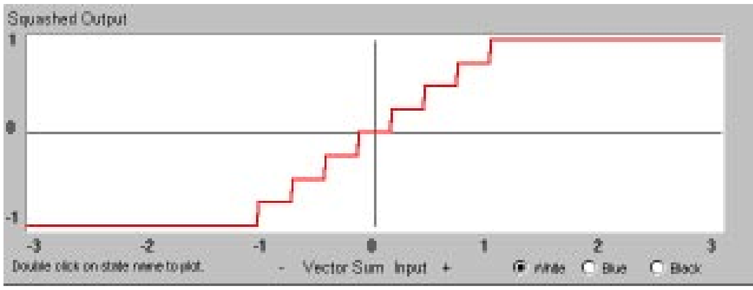
$A = 4/7$ if $(S + B)G < .33$ and ≥ 0.0

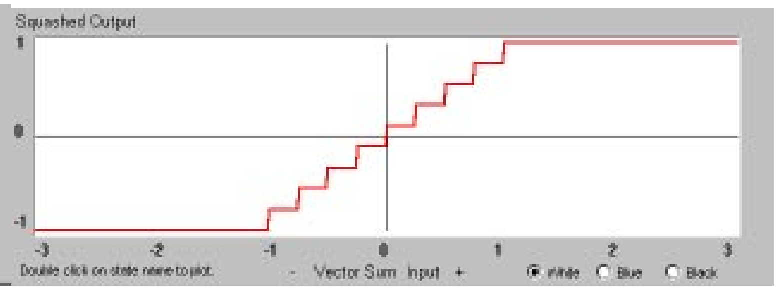
$A = 3/7$ if $(S + B)G < 0.0$ and ≥ -0.33

$A = 2/7$ if $(S + B)G < -0.33$ and ≥ -0.66

$A = 1/7$ if $(S + B)G < -0.66$ and > -1.0

$A = 0.0$ if $(S + B)G < -1.0$





Neuvary Function:

$A = 1.0$ if $(S+B)G > 1.0$

$A = .875$ if $(S+B)G < 1.0$ and $\geq 5/7$

$A = 0.75$ if $(S+B)G < 5/7$ and $\geq 3/7$

$A = 0.625$ if $(S+B)G < 3/7$ and $\geq 1/7$

$A = 0.5$ if $(S+B)G < 1/7$ and $\geq -1/7$

$A = 0.375$ if $(S+B)G < -1/7$ and $\geq -3/7$

$A = 0.25$ if $(S+B)G < -3/7$ and $\geq -5/7$

$A = 0.125$ if $(S+B)G < -5/7$ and > -1.0

$A = 0.0$ if $(S+B)G < -1.0$

Decary Function:

$A = 1.0$ if $(S+B)G \geq 1.0$

$A = 8/9$ if $(S+B)G < 1.0$ and $\geq .75$

$A = 7/9$ if $(S+B)G < 0.75$ and ≥ 0.5

$A = 6/9$ if $(S+B)G < 0.5$ and ≥ 0.25

$A = 5/9$ if $(S+B)G < 0.25$ and ≥ 0.0

$A = 4/9$ if $(S+B)G < 0.0$ and ≥ -0.25

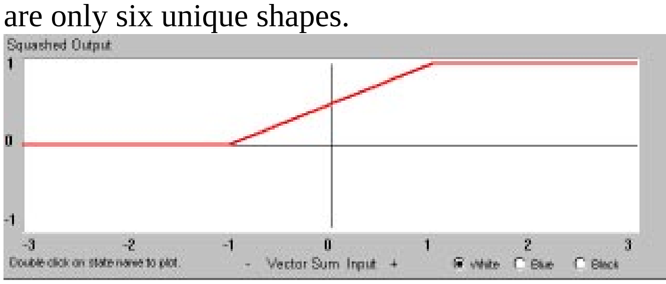
$A = 3/9$ if $(S+B)G < -0.25$ and ≥ -0.5

$A = 2/9$ if $(S+B)G < -0.5$ and ≥ -0.75

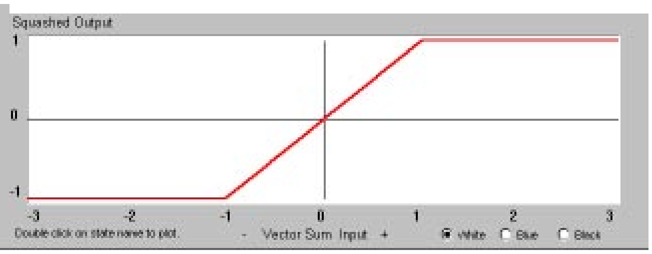
$A = 1/9$ if $(S+B)G < -0.75$ and > -1.0

$A = 0.0$ if $(S+B)G < -1.0$

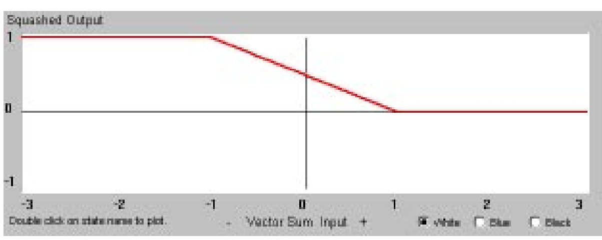
Based on the above shape called Linear, here are the eight variations of Bipolar, Invert and Negate. As you can see there are only six unique shapes.



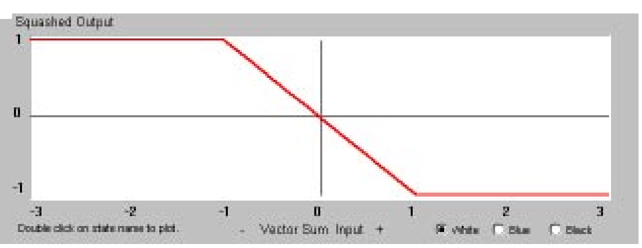
Unipolar, Not Reversed, Not Negated



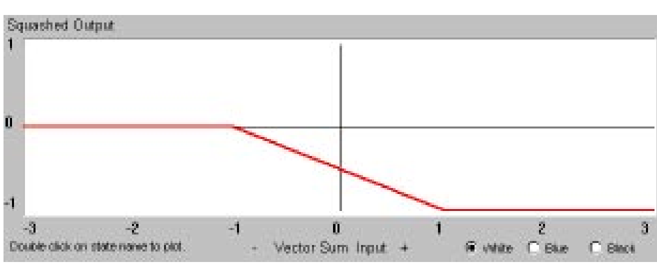
Bipolar, Not Reversed, Not Negated



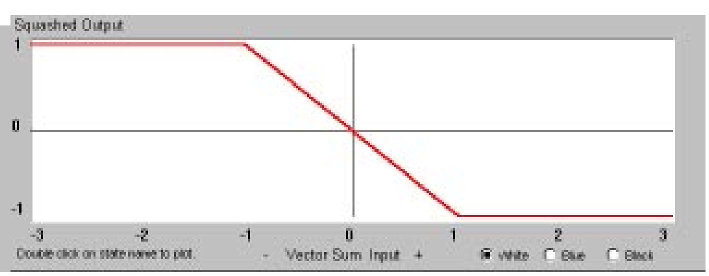
Unipolar, Reversed, Not Negated



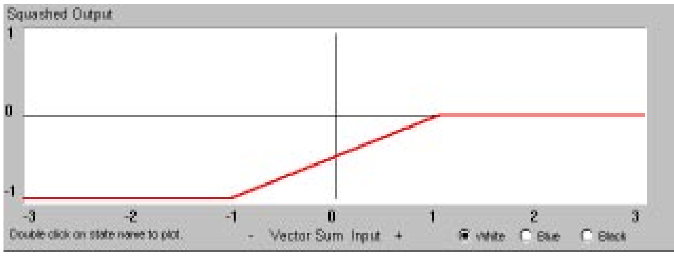
Bipolar, Reversed, Not Negated

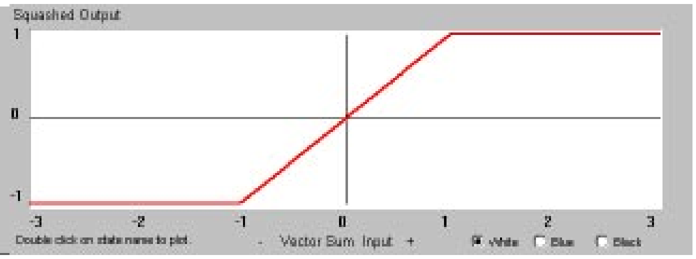


Unipolar, Not Reversed, Negated



Bipolar, Not Reversed, Negated

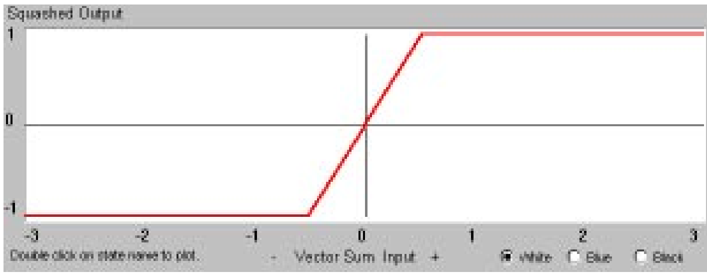




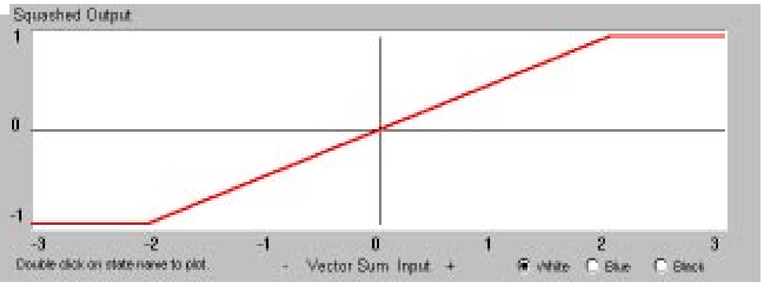
Unipolar, Reversed, Negated

Bipolar, Reversed, Negated

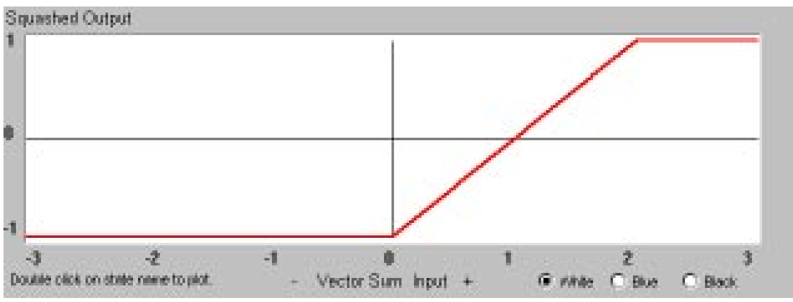
Here are pictures showing a Gain of 2, a Gain of .5, a Bias of -1 and a Bias of +1.



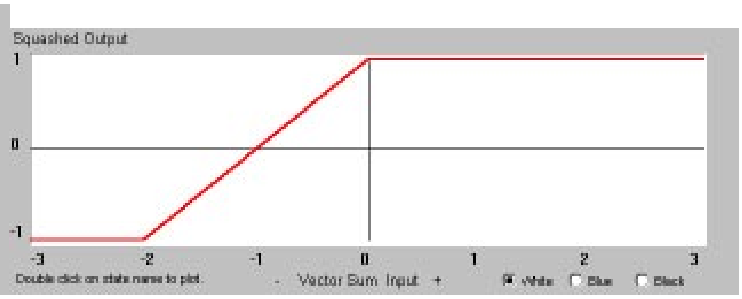
Gain (Gain times Multiplier) of 2.0



Gain (Gain times Multiplier) of 0.5

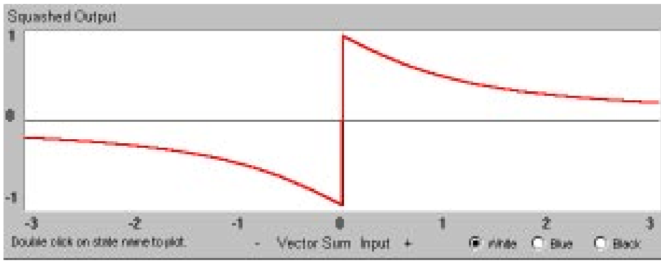


Bias of -1.0

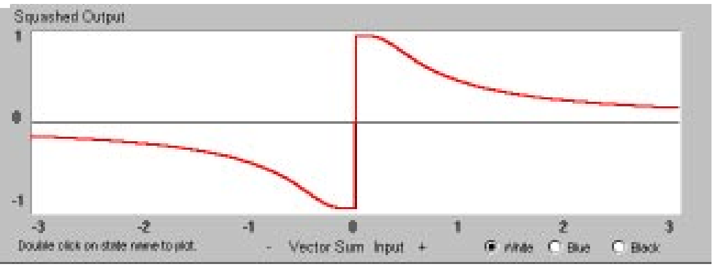


Bias of +1.0

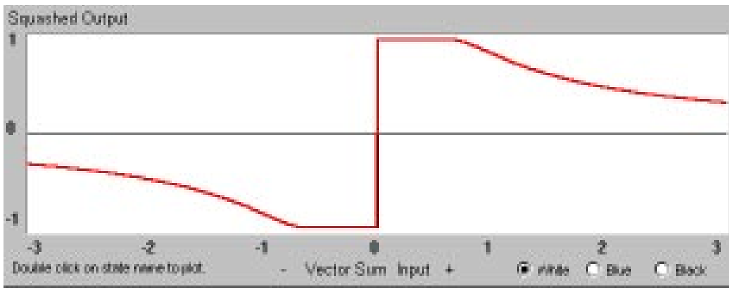
As a final variation the input may be inverted by dividing into 1. This produces an expanding effect rather than the squashing as the previous. Here are the various functions with the input inverted. Inversion of the output value does not make sense as it forces the output value outside of the +/- 1 limits over all values. The input inversion only needs clipping to the +/-1 limits near zero input.



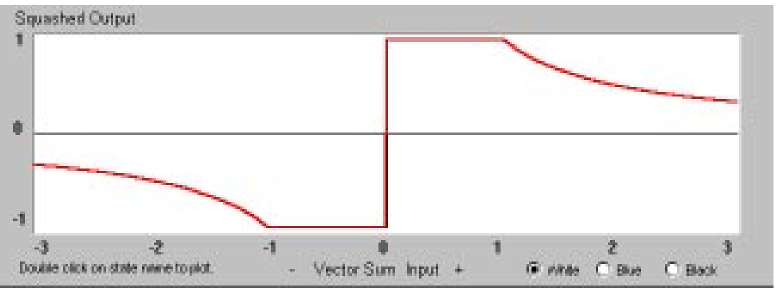
Inverted ArcTangent



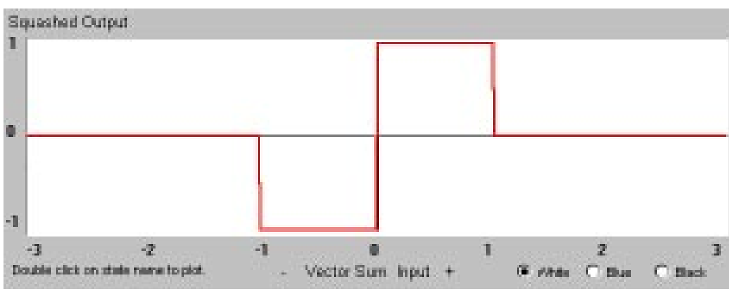
Inverted Logistic



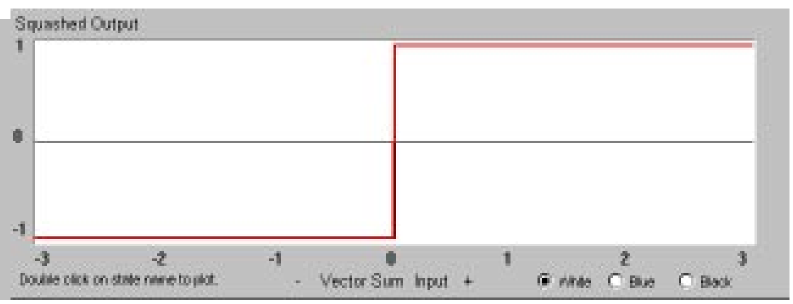
Inverted Sine



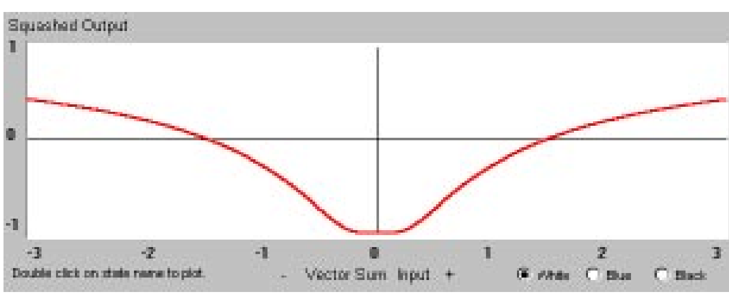
Inverted Linear



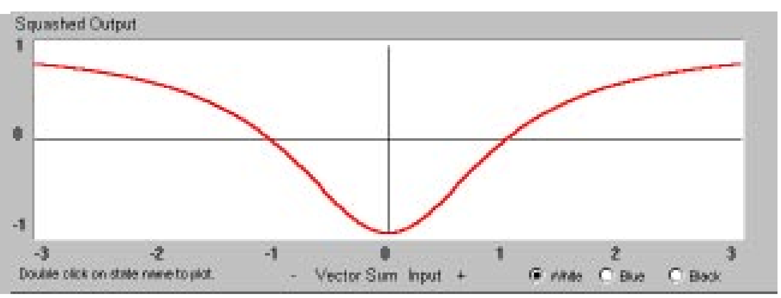
Inverted Trinary



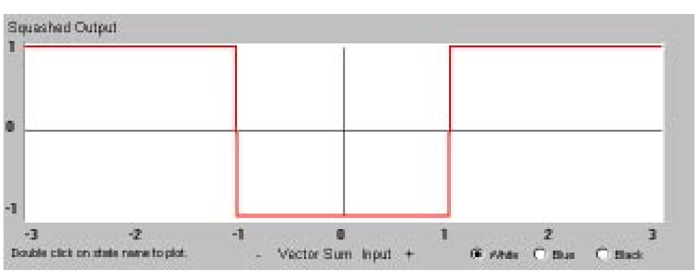
Inverted Binary



Inverted Gausian

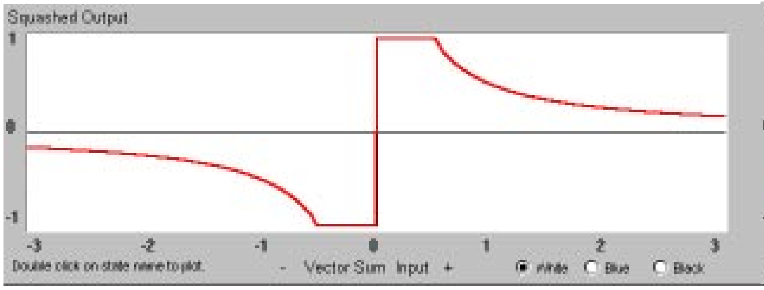


Inverted Cauchy

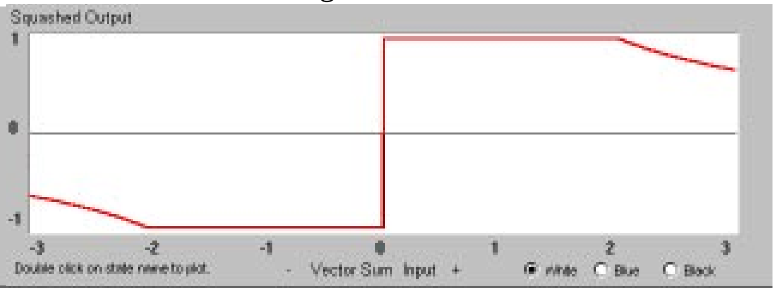


Inverted Slot

Of course with inversion the gain effect is also inverted. The bias effect is not changed.



Inverted Linear Gain = .5



Inverted Linear Gain = 2

What does it mean?

- Ad hoc
- Modeling structure
- Static Platform
- Dynamic Platform

Fuzzy Cognitive Maps are completely ad hoc. So they mean pretty much what you want it them to mean. The purpose of any modeling structure of this sort is to provide a platform for exploration. Fuzzy Cognitive Maps are dynamic platforms. They have their own internal rules and constraints. Within those rules and constraints it is subject to external influence and is dynamic. The claim in the literature is that the Fuzzy Cognitive Map form of dynamic modeling is easily set up (influenced) to represent any story. The story may be in printed or in mind, it is easily placed into the connected graph of the Fuzzy Cognitive Map. The dynamics of the Fuzzy Cognitive Map then play that story out in an iteration with feedback manner. It may not be how reality works, but it does behave in potentially useful ways. Text constructs have similar meanings within their constraints.

In the spirit of the verbal labels we have become accustomed to affixing to membership functions we can also affix verbal labels to squashing functions. Where the membership function might be identified as “very small” or “medium big” the various squashing functions may be verbally labeled. Here are some examples with possible verbal labels.

Squashing Function	Verbal Label	Meaning
high gain	narrowly	Very active causal connections
low gain	broadly	Minor causal connections
no bias	centered	Activity is in the middle
negative bias	right	Activity is up scale
stepped	clumped	Histogram like aggregation
binary	crisp	Bang-bang Causal Activity
inverted	dispersed	Things get spread out
linear	directly proportional	Easily understood (within limits)
linear inverse	inversely proportional	Also easily understood as backwards
slotted	go-no go	Here it’s OK, there it is not

Conclusion

The claim here is that as we have accepted the membership function we can also accept the squashing function as a useful fuzzy tool.